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Canonical form

For ~~heat~~ weave eqn $u_{tt} - c^2 u_{xx} = 0$.

The characteristic eqn is $\xi = x + ct$, $\eta = x - ct$

The corresponding canonical form is $u_{\xi\eta} = 0$ $\left(\begin{array}{l} u_{xx} = u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta} \\ u_{tt} = c^2(u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta}) \end{array} \right)$

Now solve $u_{\xi\eta} = 0$.

$\Rightarrow u_{\xi\eta} = \frac{\partial^2 u}{\partial\xi\partial\eta} = 0$

$\Rightarrow \frac{\partial}{\partial\xi}\left(\frac{\partial u}{\partial\eta}\right) = 0$

$\Rightarrow \frac{\partial u}{\partial\eta} = \phi(\eta)$ (where ϕ is arbitary function.)

$\Rightarrow u(\xi,\eta) = \int \phi(\eta)\,d\eta + g(\xi)$ where g is a

$\Rightarrow u(\xi,\eta) = f(\eta) + g(\xi)$ arbitar of ξ

$\Rightarrow u(\xi,\eta) = f(\eta) + g(\xi)$

$\Rightarrow u(\xi,\eta) = \varphi(\xi) + \psi(\eta)$

$\Rightarrow u(\xi,\eta) = \varphi(x+ct) + \psi(x-ct)$

$\Rightarrow \boxed{u(x,t) = \varphi(x+ct) + \psi(x-ct)}$

where φ & ψ arbitary function.

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Find the canonical form of

$u_{xx} + (2\operatorname{cosec} y) u_{xy} + (\operatorname{cosec}^2 y) u_{yy} = 0$

$\Rightarrow$ Comparing the given eqn

$A u_{xx} + B u_{xy} + C u_{yy} = F(x, y, u_x, u_y)$

$\therefore A = 1, \; B = 2\operatorname{cosec} y \quad C = \operatorname{cosec}^2 y.$

Then $B^2 - 4AC$

$= (2\operatorname{cosec} y)^2 - 4 \cdot 1 \cdot \operatorname{cosec}^2 y$

$= 0$

$\therefore$ Then given PDE is parabola.

$\therefore$ consider the eqn —

$A\lambda^2 + B\lambda + C = 0$ where $\lambda = -\frac{dy}{dx}$

$\therefore \lambda^2 + (2\operatorname{cosec} y)\lambda + \operatorname{cosec}^2 y = 0$

$\therefore \lambda = -\operatorname{cosec} y$

characteristic eqn

$\therefore -\frac{dy}{dx} = -\operatorname{cosec} y$

$\therefore \frac{dy}{dx} = \operatorname{cosec} y$

$\therefore \frac{dy}{\operatorname{cosec} y} = dx$

$\therefore \sin y \, dy = dx.$

$\therefore -\cos y = x + c_1$

$\therefore x + \cos y = c_2$

$\therefore \xi = x + \cos y.$ Let $\eta = y$

$$J(x,y) = \begin{vmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{vmatrix} = \begin{vmatrix} 1 & -\sin y \\ 0 & 1 \end{vmatrix} = 1 \neq 0.$$

General Solutions:

Prob. Find the general solution of

$x^2 u_{xx} + 2xy\, u_{xy} + y^2 u_{yy} = 0.$

$\Rightarrow$ Comparing the given eqn with

$A u_{xx} + B u_{xy} + C u_{yy} = 0$

$A = x^2 \quad B = 2xy \quad C = y^2$

$\therefore$ The $B^2 - 4AC = 4x^2y^2 - 4x^2y^2 = 0.$

so given eqn is parabolic type.

$\therefore$ Then $A\lambda^2 + B\lambda + C = 0$

$\Rightarrow x^2\lambda^2 + 2xy\lambda + y^2 = 0.$

$\Rightarrow \lambda = \dfrac{-2xy + \sqrt{0}}{2x^2}$

$\Rightarrow \lambda = -\dfrac{\cancel{2}xy}{\cancel{2}x^2} = -\dfrac{y}{x}.$

$\therefore$ characteristic eqn

$\dfrac{dy}{dx} = +\dfrac{y}{x}.$

$\therefore \dfrac{dy}{y} = \dfrac{dx}{x}.$

$\therefore \log y = \log x + \log c$

$\therefore \log y - \log x = \log c.$

$\therefore \log \dfrac{y}{x} = \log c.$

$\therefore \dfrac{y}{x} = c.$

$\therefore$ Let $\xi = \dfrac{y}{x}.$ Let $\eta = y$

Then $J(\xi, \eta) = \begin{vmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{vmatrix} = \begin{vmatrix} -\frac{y}{x^2} & \frac{1}{x} \\ 0 & 1 \end{vmatrix} = -\dfrac{y}{x^2} \neq 0$ ($y \neq 0$, $x \neq 0$)

$\therefore$ Then the canonical form is $u_{\eta\eta} = 0$, for $y \neq 0$ (HW)

$\therefore u_\eta(\xi, \eta) = f(\xi)$

$\therefore u(\xi, \eta) = \eta f(\xi) + g(\xi).$ where f, g are arbitrary function.

$$u(\xi, \eta) = \eta f(\xi) + g(\xi)$$
$$= x f\left(\frac{y}{x}\right) + g\left(\frac{y}{x}\right)$$
$$\therefore u(x, y) = x f\left(\frac{y}{x}\right) + g\left(\frac{y}{x}\right)$$
This is the Sol^n of PDE.

(1) Determine the general sol^n of

$4u_{xx} + 5u_{xy} + u_{yy} + u_x + u_y = 2$

(2) Obtain the general sol^n of

1. $3u_{xx} + 10u_{xy} + 3u_{yy} = 0$.
2. $y\,u_{xx} + 3y\,u_{xy} + 3u_x = 0$
3. $u_{xx} + 2u_{xy} + u_{yy} = 0$
4. $u_{xx} + 2u_{xy} + 5u_{yy} + u_x = 0$

V.V